Acta Adriat., 23 (1/2): 39—46 (1982)

YU ISSN: 0001—5113
AADRAY

FICKIAN DIFFUSION BY RANDOM SIMPLE WAVES

DIFUZIJA IZAZVANA STOHASTIČKIM VALOM U HOMOGENOM STRUJNOM POLJU

Mario Bone

Institute of Oceanography and Fisheries, Split

Fickian diffusion caused by velocity fluctuations that can be described as a stochastic wave is examined. It is demonstrated, if some asumptions on stochastic variables of Eulerian velocity and particle displacement hold, that only waves with phase velocity less than standard deviations of velocity fluctuations can produce the Fickian diffusion. The result is disscused on the examples of horizontal mixing in the sea.

INTRODUCTION

The Landau similarity theory of turbulence assumes that in turbulent mixing the velocity fluctuations on the scale of the dimension of the region where the motion is observed play the fundamental role (Landau and Lifšic, 1965, p. 135). If L is characteristic length scale and U characteristic amplitude of velocity fluctuations, than the turbulent diffusion coefficient K is $K = UL$ (Landau and Lifšic, 1965. p. 234). This gives us the possibility to describe the turbulent mixing with a simple chinematic model taking into account only the main feature of velocity fluctuations. In this paper the velocity fluctuations are described as a random simple waves. If some assumptions on stochastic variable of Eulerian velocity and particles displacement hold, it is demonstrated, that only waves with phase velocity less then standard deviation of velocity fluctuations produce the Fickian diffusion. This is clear because the fast waves will make the fluid particle to oscillate around its initial position and there will be no mixing. If the phase velocity is negligible compared to standard deviation of velocity fluctuations, the given relation for diffusion coefficient is obtained. The result is disscused on the examples of horizontal mixing in the sea.

DEFINITIONS

We will consider the Eulerian velocity field as stochastic process with parameters $(\vec{X}, t)$ where $\vec{X}$ is radius vector of space point and t time. Let it be

(1) $\quad \vec{u}\,(\vec{X}, t) : \Omega \rightarrow R^3,$

$\vec{u}(\vec{X}, t)$ denotes maping of sample set Ω in R^3 state space of $\vec{u}$ at given space pont $\vec{X}$ and time t. Without defining physical meaning of Ω we will assume that $\vec{u}$ is stationary in space and time and ergodic. In this case the distribution function of $\vec{u}$, $F_u[\vec{u}(X, t)]$, is defined from the behaviour of stochastic variable $\vec{u}$ in time or space.

The particle displacement $\vec{Y}(\vec{X}, t)$ from $\vec{X}$ and in time interval [o, t] can be defined, once defined $\vec{u}$, as a solution of integral equation

$$(2) \qquad \vec{Y}(\vec{X},t) = \int_o^t \vec{u}\,[\vec{X} + \vec{Y}(\vec{X}, \zeta), \zeta]\, d\zeta$$

If defined stochastic process $\vec{Y}(\vec{X}, t)$, from equation (2), assuming the existence of integral, than it is the maping of the same sample set Ω as for $\vec{u}$ in the state space R^3. Let the distribution function of stochastic process $\vec{Y}(\vec{X}, t)$ be $F_y\,[\vec{Y}(\vec{X}, t)]$. Essentialy it is transition probability given $\vec{X}$ at t = o to $\vec{X} + \vec{Y}$ at t.

The Lagrangian velocity field we define as

$$(3) \qquad \vec{v}(\vec{X}, t) = \vec{u}\,[\vec{X} + \vec{Y}(X, t), t]$$

From equation (3) follows that sample set Ω of stochastic process $\vec{v}(\vec{X}, t)$ is the same as for stochastic process $\vec{u}(\vec{X}, t)$.

Let $F_v[\vec{v}(\vec{X}, t)]$ be the distribution function of $\vec{v}(\vec{X}, t)$.

We see that sample set Ω is the same for stochastic variables $\vec{u}$, $\vec{Y}$ and $\vec{v}$ and we can use the theorem of conditional probability.

For all distribution functions here considered we will suppose the existence of probability density function. Applying the theorem of conditional probability from the previous definitions it follows that

$$(4) \qquad f_v\,[\vec{v}(\vec{X}, t)] = \int_Y f_Y\,[\vec{Y}(X, t)]\; f_{u|Y}\,[\vec{u}(\vec{X}+\vec{Y}, t) \mid \vec{Y}(\vec{X}, t)]\, dY$$

where $dY = dY_1\, dY_2\, dY_3$, Y is the state space of stochastic variable $\vec{Y}(\vec{X}, t)$, $f_{u|Y}$ is probability density function of $\vec{u}$ given $\vec{Y}$.

If we define stochastic process $\vec{Y}(\vec{X}, t)$ homogeneous in space, i. e. independent of $\vec{X}$, from equation (4) follows that stochastic process $\vec{v}(\vec{X}, t)$ is stationary in time and space. In this case it is

$$(5) \qquad f(\vec{v}_1 \,\&\, \vec{v}_2; t_2 - t_1) = \int_Y f_Y(\vec{Y}; t_2 - t_1) f_{u_1 \& u_2|Y}(\vec{u}_1 \,\&\, \vec{u}_2 \mid \vec{Y}; t_2 - t_1)\, dY$$

where $\vec{v}_1 = \vec{v}(\vec{X}, t_1)$, $\vec{v}_2 = \vec{v}(\vec{X}, t_2)$, $\vec{u}_1 = \vec{u}(\vec{X}, t_1,)$, $\vec{u}_2 = \vec{u}(\vec{X} + \vec{Y}, t_2)$,
& denotes »and«, and the parameters are denoted after;

If we define $\vec{Y}(t)$ as a process with independent increments from equation (2) than follows that $\vec{u}$ is stochasticaly independent of $\vec{Y}$. In this case from equation (5) we obtain relation between Lagrangian velocity correlation function $R_{ij}(t)$ and Eulerian velocity correlation function $r_{ij}(\vec{Y}, t)$

$$(6) \qquad R_{ij}(t) = \int_Y f_Y[\vec{Y}(t)] \; r_{ij}(\vec{Y}, t)\, dY$$

COSEQUENCES OF STOCHASTIC INDEPENDENCY OF EULERIAN VELOCITY COMPONENTS

The components of the Eulerian velocity could be taken as stochastically independent. In this case it will be

$$(7) \qquad r_{ij} = o \quad \text{if} \quad i \neq j$$

From equations (6) and (7) follows that

$$(8) \qquad R_{ij} = o \quad \text{if} \quad i \neq j$$

From equations (2) and (7) follows

$$(9) \qquad D_{ij} = \operatorname{cov}(Y_i Y_j) = o \quad \text{if} \quad i \neq j$$

The transition probability density function is of a the form

$$(10) \qquad f_Y(Y) = f_1(Y_1) f_2(Y_2) f_3(Y_3)$$

We can see that in the case of stochastic independency of the components of the Eulerian velocity fluctuations the statistical description of the motion can be done observing the motion along given axis independenthy of other axes. This case will be considered here and the axis subscripts in the furter text will be omitted.

RANDOM SIMPLE WAVE AND FICKIAN DIFFUSION COEFICIENT

We assume that the spectral density of the Eulerian velocity fluctuations is

$$f_{\omega\varkappa}(\omega, \varkappa) = \tfrac{1}{2}\left[\delta(\omega - \bar{\omega}) + \delta(\omega + \bar{\omega})\right] f_{\varkappa}(\varkappa) \tag{11}$$

where δ denotes Dirak's delta function, ω circular frequencies, $\pm\bar{\omega}$ characteristic circular frequencies of the considered waves and $\varkappa$ wave number. According to Bochner's theorem (e.g. Welsh, 1970, p. 526) the covariance of Eulerian velocity will be

$$r(y, \zeta) = \cos\bar{\omega}\zeta \int_{\infty}^{\infty} e^{i\varkappa y} f_{\varkappa}(\varkappa)\, d\varkappa \tag{12}$$

By inserting it into equation (6) for the covariance of

Lagrangian velocity after interchange the order of integration we obtain

$$R(\zeta) = \cos\bar{\omega}\zeta \int_{\infty}^{\infty} \Phi(\varkappa;\zeta)\, f_{\varkappa}(\varkappa)\, d\varkappa \tag{13}$$

where $\Phi(\varkappa; \zeta)$ is the characteristic function of $f_y(y; t)$.

Assuming the existence of integrals

$$T = \int_{0}^{\infty} R(\zeta)\, d\zeta \tag{14}$$

where T is Lagrangian integral time scale, and

$$S = \int_{0}^{\infty} \zeta R(\zeta)\, d\zeta \tag{15}$$

for $t > T$ the displacement covariance according to Taylor's equation which could be obtained from equations (2), (3) and (14) asymptoticaly will be

$$D(t) = 2U^2\, Tt \tag{16}$$

and the transition probality density will be Gaussian

$$f_{Y}(Y; t) = (4\pi Kt)^{-1/2} \exp(-Y^2/4Kt) \tag{17}$$

where the Fickian diffusion coefficient K is (e.g. Monin & Yaglom, 1977, p. 608)

$$K = U^2\, T \tag{18}$$

The characteristic function of the transition probability density function is (.eg. Zelen & Severa, 1968)

$$\Phi(\varkappa; t) = \exp(-K\varkappa^2 t) \tag{19}$$

and from equations (13) and (14) the Lagrangian integral time scale is

$$T = \int_{\infty}^{\infty} K^2/(K^2\varkappa^4+\omega^2)\, f_{\varkappa}(\varkappa)\, d\varkappa, \tag{20}$$

and from equation (18) and (20) eliminating T we obtain

$$U^{-2} = \int_{\infty}^{\infty} K\varkappa^2/(K^2\varkappa^4+\bar{\omega}^2)\, f_{\varkappa}(\varkappa)\, d\varkappa \tag{21}$$

From equation (21), providing that U, $\bar{\omega}$ and $f_\varkappa(\varkappa)$ are known, the diffusion coefficient K is defined. Assuming that

$$f_{\varkappa}(\varkappa) = \tfrac{1}{2}\left[\delta(\varkappa-\bar{\varkappa}) + \delta(\varkappa+\bar{\varkappa})\right] \tag{22}$$

i.e. by limiting the consideration to a simple wave of given frequency $\bar{\omega}$ and wave number $\bar{\varkappa}$ and regarding as random its phases and amplitudes only we obtain for the diffusion coefficient

$$K = (U^2 - C^2)^{-1/2}/\bar{\varkappa}, \quad C = \bar{\omega}/\bar{\varkappa} \tag{23}$$

We can see that the diffusion coefficient will be real only for the phase velocity C less than the standard deviation of velocity fluctuations U. The result we can explain in this way: if the phase velocity is much greater than the velocity fluctuations it will be approximatly $u = v$ with very little difference in time. In this case we have not diffusion in here considered case of simple vawe because the Lagrangian time scale is not defined, and the particles will oscillate only arround its initial positions. In the case when the phase velocity is negligible we obtain

$$K = U/\bar{\varkappa} \tag{2}$$

This result is known from similarity theory of turbulence (Landau & Lifšic, 1965, p. 234).

EXAMPLES OF HORIZONTAL MIXING IN THE SEA

The previous result we shall try to illustrate by few examples of horizontal mixing in the sea.

a) Tidal wave

The fig. 1 shows the time power spectrum of current field in the Vir sea (Zore-Armanda, Bone & Gačić, 1978). It can be observed that for the circular frequency of 2 cicle/day, corresponding to M2 component of tidal wave, spectrum shows strong maximum. This frequency could be taken as characteristic frequency of the tidal wave generating diffusion. The order of

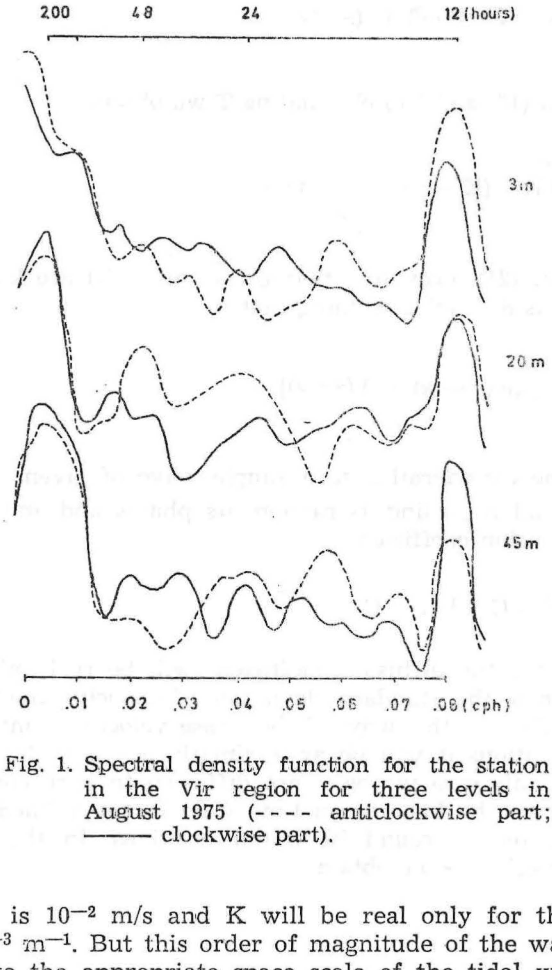

Fig. 1. Spectral density function for the station in the Vir region for three levels in August 1975 (- - - - anticlockwise part; —— clockwise part).

magnitude of U is 10^{-2} m/s and K will be real only for the wave number greater than 10^{-3} m^{-1}. But this order of magnitude of the wave number does not correspond to the appropriate space scale of the tidal wave and we can conclude that tidal vawe is fast in comparasion to particles velocity and does not generate Fickian diffusion. The same conclusion we can draw for all motion which can be described with a solution of linearised hydrodynamical equation. It is because, in the basis of linearisation of hydrodynamical equations, there is the assumption that phase velocity is much greater than particle velocity (e.g. Bone, 1977). Only the statistics on advection terms produced the diffusion processes.

b) Ocean residual current

For the ocean residual current K/U is of the order of magnitude 10^5 m (Proudman, 1952) which corresponds (from equation 24) to $\bar{\varkappa}$ of the order of magnitude of 10^{-5} m^{-1}, i.e. to the space scale of 10^3 Km. The order of magnitude of U is 10^{-1} m/s and this must be greater than $\bar{\omega}/\bar{\varkappa}$, i.e. the period of flucuations must be greater than 70 days. In this case we cauld try to describe the ocean diffusion with a wave which have the space dimension of 10^3 km and seasonal period of fluctuations.

c) Residual current in the bay

For the example we take the Kaštela bay. The space scale of the bay is 10^4 m. The diffusion coefficient in the residual current is determined from the field of suspended particles (Gačić, 1976). The order of magnitude of diffusion coefficient is 10 m^2/s. The standard deviation of velocity fluctuations in the residual current is of the order of magnitude 10^{-2} m/s. In this case $\bar{\varkappa}$ is of order of magnitude 10^{-3} m^{-1} and $\bar{\omega}$ could have tidal or less frequencies. We could try to describe the diffusion by the wave motion induced by tide having the dimension of the bay and tidal frequencies.

REFERENCES

Bone, M. 1977. On neglecting the acceleratin advective term in the hydrodynamical equations. Acta Adriat., 17 (11): 1—8 p.

Gačić, M. 1975. Ekstincija svijetlosti u morskoj vodi Kaštelanskog zaljeva. Hidrografski godišnjak 1973, 86—106 p.

Landau, L. D. & E. M. Lifšic, 1965. Mehanika neprekidnih sredina. Građevinska knjiga, Beograd. 743 p.

Monin, A. S. & A. M. Yaglom. 1977. Statistical fluid mechanic: mechanic of turbulence. The Mit Press, Cambridge, Massachusetts and London, England. 769 p.

Proudman, J. 1952. Dynamical oceanography. Methuen & Co. LTD., London, John Wiley & Sons, Inc., New York, 409 p.

Welsh, D. J. A. 1970. Probability theory and its application in Mathematics applied to physics, Editor E. Roubine. Springer Verlag, Berlin, Heidelberg, New York, UNESCO, Paris, 65—561 p.

Zelen, M. & N. C. Severo. 1968. Probability functions in Handbook of mathematical fujction, Editor M. Abramowitz & I. A. Segun. Dover Publication, Inc., New York, 925—996 p.

Zore-Armanda, M., M. Bone and M. Gačić. 1979. Some dynamic characteristic of the East Adriatic coastal area. Acta Adriat., 19 (10): 83—102 p.

DIFUZIJA IZAZVANA STOHASTIČKIM VALOM U HOMOGENOM STRUJNOM POLJU

Mario Bone
Institut za oceanografiju i ribarstvo, Split

KRATKI SADRŽAJ

Fikova difuzija promatrana je kao rezultat fluktuacija brzina u homogenom i stacionarnom strujnom polju koje se mogu opisati kao stohastički val date frekvence dok su njegove amplitude i valni brojevi slučajne veličine. Rezultat je vrlo jednostavna relacija između koeficijenta Fikove difuzije K, standardne devijacije fluktuacija brzina U i karakterističnog valnog broja fluktuacija na promatranoj prostornoj skali L: K = U/L. Ovaj rezultat poznat je iz Kolmogorove teorije turbulencije s tim što je pokazano da je Fikova difuzija u ovdje promatranom slučaju moguća samo ako su fazne brzine promatranog stohastičkog vala male prema standardnoj devijaciji fluktuacija brzina. U kratkoj diskusiji iznijeti su neki primjeri iz horizontalnog mješanja u moru.