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ON SEMIGRAPHIC ESTIMATION OF PARAMETERS OF GOMPERTZ FUNCTION AND ITS APPLICATION ON FISH GROWTH

O SEMIGRAFIČKOM ODREĐIVANJU PARAMETARA GOMPERTZOVE FUNKCIJE I NJENOJ PRIMJENI NA RAST RIBA

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A method of estimation of parameters of Gompertz function is given. Thus, we tried to estimate, on the basis of distribution of numerical values, the range within the real value of asymptote may be. Using *ln ln* or *log log* transformations the values of Y subtracted from the estimated values of asymptote are plotted against the relevant values of X. The estimated value which gives the best linear fit is equal or approximate to the real value of asymptote.

The method of estimation is shown on two examples of fish growth.

INTRODUCTION

The investigations of the growth in length of fish in function of time are indispensable in fishery biology since they provide the basis for the further studies of mortality, population dynamics in general, stock assessment, regulation of fishing, etc. In doing this, if the data on growth are to be further by used, they should be mathematically approximated by relevant functions.

Von Bertalanffy equation of growth is mainly used now as the basis for the further computations. This equation is recommended in the manuals for fish stock assessments as well (Gulland, 1965). However, the distribution of data on fish lengths in relation to time very often shows the initial stage of slower growth, followed by the stage of accelerated growth and finally the stage of deceleration. In addition, the first two stages usually cover the first third of the total time needed for attaining the maximum length. That is to say, the growth curve is often of asymmetric sigmoid shape which may not be sufficiently approximated by von Bertalanffy function.

However, G o m p e r t z function is one of the functions by which the asymmteric sigmoid curve of fish growth may be described. This function may be vritten in two basic forms. They are:

$$l_t = a \cdot e^{-be^{-ct}} \quad (1) \text{ or}$$

$$l_t = a \cdot b^{c^t} \quad (2),$$

where l_t is the fish length in time t, *a* is the asymptote, *b* and *c* are constants and *t* is time.

However, as far as this function is concerned, some difficulties occur in techniques of estimation of its parameters. Thus, B r o d y (1945), according to R i f f e n b u r g h (1960), proposes the technique of serial expansions. He holds that owing to this the function is of little practical use. Other methods of estimation of parameters of function of form (2) were developed after. However, they are limited in a number of ways. The deficiency of S c h u l e r's method, according to R i f f e n b u r g h (1960), is that the number of observed numerical values should be *3n*, where *n* should be a positive integer and the values of independent variable should be integers with the equal itervals between them. The deficiency of R i f f e n b u r g h's (1960) method is that the numbers on abscisa should be integers. To determine the parameters of G o m p e r t z function of the horm (1) for the length growth of the northern anchovy (*Engraulis mordax* G i r a r d), K r a m e r and Z w e i f e l (1970) had to calculate the length — weight relationship in addition to the data on length. S a k a g a w a and K i m u r a (1976) estimate the parameters of G o m p e r t z growth curve for the same species using the method of M a r q u a r d t's algorithm (C o n w a y et al., 1970) which does not include any of the mentioned limitations, but is rather complicated mathematically.

The aim of this paper is to propose a relatively simple method of numerical estimation of G o m p e r t z function parameters which is subject to none of the mentioned limitations.

SEMIGRAPHIC METHOD

A method of determination of G o m p e r t z function of form (1) is going to be described here in detail.

Taking the logarithm of this equation it is obtained that:

$$\ln l_t = \ln a - be^{-ct} \quad (2).$$

If the equation (2) is rearranged thus that *ln a* is shifted to the left side and its logarithm taken once again it is obatined that:

$$\ln(\ln a - \ln l_t) = \ln b - ct \quad (3),$$

i. e. the linear equation where:

$$X = t;\ Y = \ln(\ln a - \ln l_t)$$

Since the value of *a* is still unknown, this operation may appear not to bring the solution closer. However, on the basis of the actual data on fish lengths (l_{t_i}) in relation to time (t_i) it is possible to estimate the values of asymptote (*a*) to a certain extent. If the estimated value (a_e) is equal or approximate to the real value of asymptote (a_r), the values ln (ln a_e -ln l_{t_i}) plotted against the time (t_i) should, according to the equttion (3), lie on the straight line:

$$Y = \ln b - ct \qquad (3').$$

If:

$$a_e < a_r$$

the plotted values will decline from the straight line in negative, and if

$$a_e > a_r$$

they will decline in positive direction.

This may be illustrated by the following example. Gompertz equation with arbitrarily chosen parameters is taken:

$$l_{t_i} = 120\, e^{-0.485e^{-0.238t_i}} \qquad (4),$$

upon which the valeus l_{t_i}; for a definite number of t_i were calculated (Table 1).

Table 1. Values of l_{t_i} calculated for the relevant t_i values on the basis of the equation (4).

t_i	0	0.5	1	1.5	2	2.5	3	3.5	4
l_{t_i}	73.88	78.02	81.88	85.46	88.78	91.83	94.63	97.19	99.51

The estimated values (a_e) were taken thereupon as follows:

$$110 = a_e < a_r \qquad a_e = a_r = 120 \qquad 140 = a_e > a_r$$

and ln (ln a — ln l_{t_i}) values were separately calculated for each a_e value. The distribution of ln (ln a — ln l_{t_i}) values in relation to time (t_i) is given in Fig. 1.

It is obvious that in this way the approximate values of asymptote may be estimated with satisfactory exactness. In practice, however, it often happens that the dispersion of data on length is of large extent. In these cases it will be difficult to conclude which of the estimated values of a_e gives the best linear fit. In such a cases it is most convenient to calculate the linear correlation coefficients for the pairs:

$$X_i = t_i;\ Y_i = \ln(\ln a_e - \ln l_{t_i}),$$

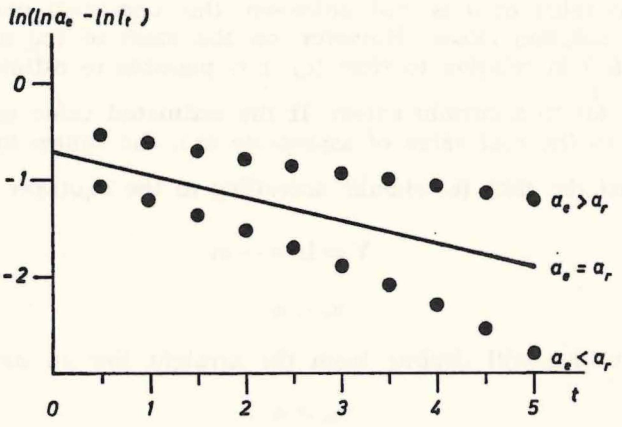

Fig. 1. Distribution of values of $ln\,(ln\,a_e - ln\,l_t)$ at differently estimated values of asymptote.

for every estimated value of asymptote separately. In the given example the distribution of correlation coeficients was as follows:

a_e	110	120	140
r	—0.996	—1.000	—0.998

Accordingly, in estimation of the a_r value, going from the underestimated to the overestimated values of a_e, the correlation coefficients, inasmuch as the data on growth may be approximated by Gompertz function, will first increase and than decrease. The highest correlation coefficient will represent the value of a_e which is equal or approximate to the value of a_r, i. e. to the real value of asymptote.

When the approximate value of asymptote (a) is estimated in described manner, the values of constants *c* and *b* are determinated from equation (3) by the method of linear regression where:

$$c = \frac{\sum_{i=1}^{n} (X_i - \bar{X})\; Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \qquad (5),$$

while

$$\ln b = \bar{Y} - c\bar{X} \qquad (6),$$

where, as shown earlier:

$$X_i = t_i;\; Y_i = \ln(\ln a - \ln l_t)_i;\; n = \text{total number of observations.}$$

The parameters of Gompertz function of the from (2) may be determined in the same manner, according to the following tranformation:

$$\log\left(\log\frac{a_e}{l_t}\right) = \log\left(\log\frac{1}{b}\right) - t\log c \qquad (7).$$

First example

In the above mentioned paper Riffenburgh demonstrates the way of determination of parameters of Gompertz function of the form (2) using the length-age data on yellowfin tuna *(Neothunnus macropterus)* given by Moore (1951).

This example will be applied here, as well. Ages and modal lengths of tuna are given in Table 2.

Table 2. Modal lengths and estimated age of yellowfin tuna *(Neothunnus macropteurs)* in the Hawaiian waters

t(months)	l (cm)	t(months)	l (cm)
1	47.2	27	138.7
2	47.2	28	138.7
9	69.7	29	140.3
12	93.0	30	138.7
13	93.0	31	146.3
20	125.3	34	156.4
21	120.2	35	152.1
22	125.3	38	152.4
23	130.1	39	158.2
24	142.6	41	156.4
25	134.5	43	162.5
26	138.7	44	163.1
		53	167.6

According to the distribution of lengths (Fig. 2) the value of asymptote (a_r) was estimated to be somewhere between 160 and 190 cm. Therefore, for the series of values (a_e), the correlation coefficients were calculated by described method, as given in Table 3.

Table 3. Coefficients of correlation ($X = t_i$; $Y = \ln(\ln a_e - \ln l_{t_i})$) for the estimated values of asymptote (a_e) in tuna from the Hawaiian waters.

a_e	160	168	170	173	175	180	190
r	—0.9821	—0.9834	—0.9868	—0.9921	—0.9914	—0.9867	—0.9758

According to the distribution of correlation coefficients it is evident that:

$$a_{e\,(173)} \simeq a_r .$$

Therefore, this value is used as the value of asymptote. Furthermore, the values of constants *b* and *c* were calculated by the described method (equations 5 and 6). It was obtained that:

$$l_t = 173\, e^{-1.4152\, e^{-0.0711\, t}} \qquad (8).$$

The curve estimated by the use of equation (8) agrees better with the distribution of tuna lengths than the curve given by Riffenburgh (Fig. 2). After him, Gompertz equation of tuna growth for the same data, but in form (2), is:

$$l_t = 164.78 \cdot 0.17468^{0.9177^{(t+16)}} \qquad (9).$$

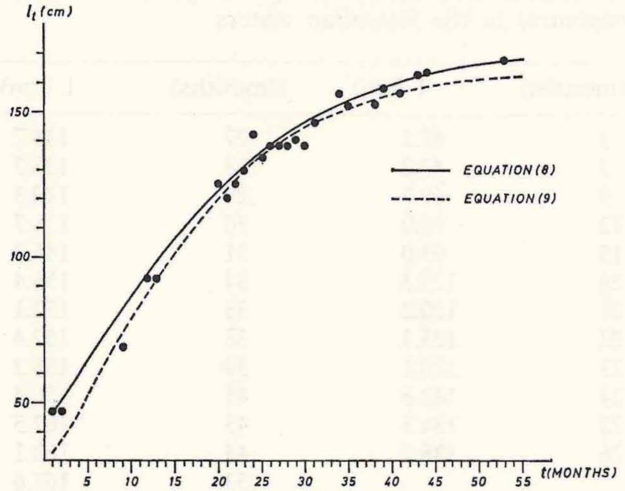

Fig. 2. The comparison of the curve of tuna growth calculated by Riffenburgh's method with the curve calculated by semigraphic method.

To testify which of these equations gives the better approximation of tuna growth, the correlation coefficients were calculated for the following pairs:

$X_i = e^{-0.0711\, t_i}$; $Y_i = \ln l_{t_i}$, for equation (8) and

$X_i = 0.9177^{(t_i + 16)}$; $Y_i = \ln l_{t_i}$, for equation (9) and it was obtained that:

$r = -0.9937$; $P < 0.001$ for equation (8) and that $r = -0.9888$; $P < 0.001$ for equation (9).

Accordingly, the method described in this paper gives the better approximation of tuna growth curve than the Riffenburgh's method.

Second example

The postlarvae of anchovy, *Engraulis encrasicolus* (L.), were reared under experimental conditions at constant temperature of 21.30°C up to the modal length (LS) of 13.01 mm (Regner, manuscript). The growth after the yolk sack resorption is given in Table 4.

Table 4. The growth of the anchovy postlarvae at 21.30°C. Modal lengths are given

t(days)	2.5	4.5	6.67	8.60	10.66	12.63	14.72
l_t(mm)	3.55	4.61	6.59	7.80	10.31	11.33	13.01

The growth of the anchovy postlarvae was first approximated by the exponential growth function:

$$l_t = a\,e^{ct} \qquad (10),$$

where l_t is the length in time *t*, *c* is the constant. It was obtained that:

$$l_t = 3.13\,e^{0.101963t} \qquad (11)$$ (Regner, manuscript).

To determine the Gompertz function on the basis of distribution of modal lengths against the time scale (Fig. 3) the asymptote (a_r) was estimated to be somewhere between 17 and 57 mm. On the basis of this the correlation coefficients were calculated according to the equation (3), as given in Table 5.

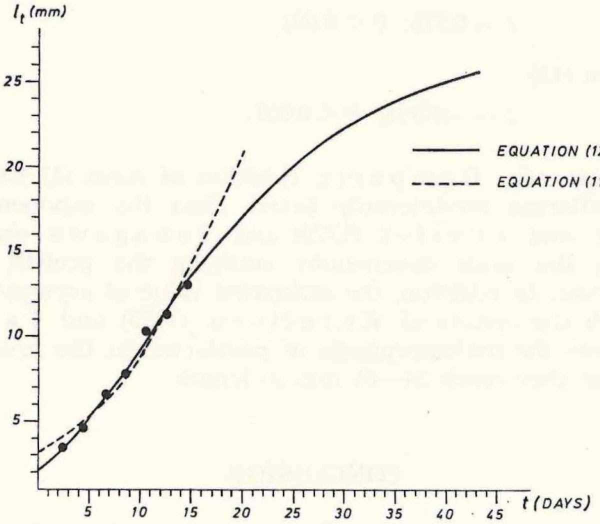

Fig. 3. Growth curves of anchovy postlarvae approximated by the exponential and by Gompertz function.

Table 5. Correlation coefficients for the estimated values of asymtote in the case of anchovy postlarvae

a_e	17	27	37	57
r	—0.99445	—0.99716	—0.9962	—0.99486

According to the distrbiution of correlation coefficients it may be concluded that:

$$a_e\,(27) \cong a_r \cdot$$

The constants *b* and *c* were calculated thereupon according to the equations (5) and (6). It was obtained for the anchovy postlarvae that:

$$l_t = 27\, e^{-2.532 e^{-0.086t}} \quad (12),$$

where l_t is the length in milimeters and *t* is time in days. As shown in Fig. 3, the modal lengths of anchovy postlarvae agree better with the curve of G o m p e r t z growth function (12) than with that of exponential (11).

The values of equation were tested here as well as in the previous example, by calculating correlation coefficients for the pairs:

$X_i = t_i;\ Y_i = \ln l_{t_i}$ for equation (11) and

$X_i = e^{-ct_i};\ Y_i = \ln l_{t_i}$ for equation (12) and it was obtained that for equation (11):

$$r = 0.978;\ P < 0.001,$$

and for equation (12):

$$r = —0.998;\ P < 0.001.$$

As it may be seen, G o m p e r t z function of form (1) fits the growth of anchovy postlarvae considerably better than the exponential function (10). K r a m e r and Z w e i f e l (1970) and S a k a g a w a and K i m u r a (1976) came to the same conclusions studying the growth of northern anchovy postlarvae. In addition, the estimated value of asymptote of 27 mm well agrees with the results of K o r n i l o v a (1955) and P a v l o v s k a i a (1955). After them the metamorphosis of postlarvae in the juvenile anchovy takes place when they reach 24—26 mm in length.

CONCLUSION

The approximate value of asymptote of G o m p e r t z function may be estimated, either graphically or calculating the correlation coefficients by the estimation of the value of asymptote on the basis of distribution of data on fish lenght against the time. In doing this, Gompertz function should be transformed into:

$$\ln(\ln a - \ln l_{t_i}) = \ln b - ct,$$

since the values $\ln(\ln a - \ln l_{t_i})$ plotted against the time, if estimated value of asymptote is close to real one, should lie on the straight line.

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KRATAK SADRŽAJ

U radu je prikazana metoda numeričkog određivanja parametara Gompertzove funkcije koja ne podliježe ni jednom od ograničenja karakterističnih za dosada opisane metode. Semigrafički način određivanja je moguć stoga što se na osnovi raspodjele izmjerenih podataka o dužinama riba u odnosu na starost može procijeniti interval unutar kojega bi se mogla nalaziti vrijednost asimptote. Kako nakon *ln ln* transformacije oblika (1) ili *log log* transformacije funkcije oblika (2), vrijednosti dužine riba oduzete od stvarne vrijednosti asimptote moraju, ucrtane u odnosu na vrijeme, ležati na pravcu, moguće je na osnovi distribucije podataka za različite procijenjene vrijednosti asimptote naći onaj niz koji je grupiran tako da se najviše približava pravcu ili pak pokazuje najveći pripadni koeficijent linearne korelacije. Vrijednost na taj način procijenjene asimptote u tome je slučaju jednaka ili bliska stvarnoj vrijednosti.

Opisani način određivanja parametara Gompertzove funkcije prikazan je u radu na dva konkretna primjera.